Rutgers University: Algebra Written Qualifying Exam

August 2017: Problem 2 Solution

Exercise. Let g be an invertible $n \times n$ complex matrix. Show that g can be written as

$$g = su = us,$$

where s is diagonalizable and all eigenvalues of u are equal to 1.

Solution.

Look at the JCF: Let $g = BJB^{-1}$ where J is the Jordan decomposition of g. Then

$$J = \begin{bmatrix} J_{\lambda_1,k_1} & & & \\ & J_{\lambda_2,k_2} & & \mathbf{0} \\ & \mathbf{0} & \ddots & \\ & & & J_{\lambda_m,k_m} \end{bmatrix}$$

Look at a single Jordan Block J_{λ_i,k_i}:

$$J_{\lambda_i,k_i} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \mathbf{0} \\ & & \ddots & \ddots \\ \mathbf{0} & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_i & & \\ & \lambda_i & \mathbf{0} \\ \mathbf{0} & \ddots & \\ & & \lambda_i \end{bmatrix}}_{\text{a diagonal matrix } D_i} \underbrace{\begin{bmatrix} 1 & \frac{1}{\lambda_i} & & \\ & 1 & \frac{1}{\lambda_i} & \mathbf{0} \\ & & \ddots & \ddots \\ \mathbf{0} & & \ddots & \frac{1}{\lambda_i} \\ & & & 1 \end{bmatrix}}_{\text{matrix } A_i \text{ with eigenvalue 1}}$$

Note: g invertible $\implies \lambda_i \neq 0 \ \forall i \implies$ we can divide by λ_i

$$\text{So, letting } D = \begin{bmatrix} D_1 & & \\ & D_2 & \mathbf{0} \\ \mathbf{0} & \ddots & \\ & & D_m \end{bmatrix} \text{ and } A = \begin{bmatrix} A_1 & & \\ & A_2 & \mathbf{0} \\ \mathbf{0} & \ddots & \\ & & A_m \end{bmatrix},$$

$$DA = AD = \begin{bmatrix} D_1A_1 & & \\ & \ddots & \mathbf{0} \\ \mathbf{0} & \ddots & \\ & & D_mA_m \end{bmatrix} = \begin{bmatrix} J_{\lambda_1,k_1} & & & \\ & J_{\lambda_2,k_2} & & \mathbf{0} \\ & \mathbf{0} & \ddots & \\ & & & J_{\lambda_m,k_m} \end{bmatrix} = J$$

$$\begin{aligned} g &= BJB^{-1} \\ &= BADB^{-1} \\ &= \underbrace{BAB^{-1}}_{=s}\underbrace{BDB^{-1}}_{=u} \\ &= su \text{ where } s = BAB^{-1} \text{ is diagonalizable and u} = BDB^{-1} \end{aligned}$$

So all eigenvalues of u are 1 since similar matrices have the same eigenvalues.
Similarly,

$$g = BJB^{-1} = BDAB^{-1} = BDB^{-1}BAB^{-1} = us$$